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N-time joint photon counting distributions for Gaussian light†

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Abstract. Results have been presented by Bedard on the derivation of the N-fold joint photocount distribution of a Gaussian (thermal) radiation field of arbitrary spectral profile, when the counting-time intervals are short compared with the coherence time of the light. In this paper we extend these results for arbitrary counting-time intervals. For simplicity we assume that the fluctuating radiation field has a fixed direction of propagation and that all the photodetector surfaces lie within one coherence area on a plane normal to this direction of propagation. We show then that the evaluation of the N-fold joint photocount distribution and the photocount moments can be reduced to the solution of either a homogeneous or an inhomogeneous (Fredholm) integral equation of the second kind. In the special case, when all N photodetectors are 'open' within the same time interval and the spectral profile of the light is Lorentzian, the N-fold joint photocount distribution and the photocount moments can be explicitly evaluated.

In the present paper we consider the case of a fluctuating electromagnetic field in the presence of N photodetectors, which register photoelectrons within finite time intervals. We shall assume for simplicity that the fluctuating radiation field is polarized and that it has a fixed direction of propagation. In addition, all the photodetector surfaces lie on a plane normal to this direction of propagation‡. Then the radiation field incident on the photodetector surfaces may be described by the random complex amplitude $V(t)$. In the following we shall also assume that this amplitude has a zero mean value.

The N-fold joint probability distribution $p(n_1, t_1, T_1; \ldots; n_N, t_N, T_N)$ that n_1 photoelectrons will be registered in the time interval t_1, t_1+T_1, etc., is given by the expression (Bedard 1967)

$$p(n_1, t_1, T_1; \ldots; n_N, t_N, T_N) = \left\langle \prod_{i=1}^{N} \frac{W_i^{n_i}}{n_i!} \exp(-W_i) \right\rangle \tag{1}$$

where

$$W_i = \alpha_i \int_{t_i}^{t_i+T_i} |V(t)|^2 \, \mathrm{d}t \tag{2}$$

$i = 1, 2, \ldots, N$. The angular brackets in equation (1) denote the appropriate statistical average with respect to the random variables W_i. The coefficient α_i in equation (2) is a measure of the quantum efficiency of the ith photodetector.

It is well known (Bedard 1967, Klauder and Sudarshan 1968) that the N-fold joint probability distribution $p(n_1, t_1, T_1; \ldots; n_N, t_N, T_N)$, as well as the generalized factorial moments, defined as

$$\langle n_1^{[l_1]} \ldots n_N^{[l_N]} \rangle = \sum_{n_1=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} \left\{ \prod_{i=1}^{N} \frac{n_i!}{(n_i - l_i)!} \right\} p(n_1, t_1, T_1; \ldots; n_N, t_N, T_N) \tag{3}$$

can be expressed in terms of the multi-dimensional generating function $G(s_1, s_2, \ldots, s_N)$

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‡ In an experimental situation the assumptions made correspond to the case where all N photodetector surfaces lie within one coherence area on a plane normal to the direction of propagation of a linearly polarized parallel beam of light.

defined by the relation

$$G(s_1, s_2, \ldots, s_N) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} \left\{\prod_{i=1}^{N}(1-s_i)^{n_i}\right\} p(n_1, t_1, T_1; \ldots; n_N, t_N, T_N). \tag{4}$$

The explicit expressions are as follows:

$$p(n_1, t_1, T_1; \ldots; n_N, t_N, T_N) = \left\{\prod_{i=1}^{N} \frac{(-1)^{n_i}}{n_i!} \frac{\partial^{n_i}}{\partial s_i^{n_i}}\right\} G(s_1, s_2, \ldots, s_N)\big|_{s_1=1, \ldots, s_N=1} \tag{5}$$

and

$$\langle n_1^{[l_1]} \ldots n_N^{[l_N]} \rangle = \left\{\prod_{i=1}^{N} (-1)^{n_i} \frac{\partial^{l_i}}{\partial s_i^{l_i}}\right\} G(s_1, s_2, \ldots, s_N)\big|_{s_1=0, \ldots, s_N=0}. \tag{6}$$

Substituting equation (1) into equation (4), we see that the generating function becomes

$$G(s_1, s_2, \ldots, s_N) = \left\langle \exp\left(-\sum_{i=1}^{N} s_i W_i\right)\right\rangle \tag{7}$$

where W_i is given by equation (2).

The relations presented up to this point apply to any radiation field. Let us assume from now on that the radiation field is of Gaussian nature, so that the N-fold joint probability distribution for the random complex amplitude $V(t)$ (of zero mean value) is given by the following expression:

$$p_N(V_1, V_2, \ldots, V_N) = \frac{|A|}{(2\pi)^N} \exp(-\tfrac{1}{2} V^{\dagger} A V). \tag{8}$$

Here V is the column matrix

$$V = \begin{pmatrix} V_1 \\ \cdot \\ \cdot \\ \cdot \\ V_N \end{pmatrix} \tag{9a}$$

with $V_i = V(t_i)$, $i = 1, 2, \ldots, N$ and

$$V^{\dagger} = (V_1^*, V_2^*, \ldots, V_N^*). \tag{9b}$$

Also $|A|$ is an $N \times N$ determinant, whose elements are defined in terms of the mutual coherence function (Mehta 1965)

$$\Gamma_{ij} = \langle V_i^* V_j \rangle \tag{10}$$

through the relation

$$2(A^{-1})_{ji} = \Gamma_{ij}. \tag{11}$$

Let us suppose now that the earliest measurement of our photodetectors starts at time $-\frac{1}{2}T$, and the latest measurement ends at time $\frac{1}{2}T$, so that all the time intervals $(t_i, t_i + T_i)$, $i = 1, 2, \ldots, N$, lie within $(-\frac{1}{2}T, \frac{1}{2}T)$. Furthermore, let us take the Karhunen-Loeve expansion (Davenport and Root 1958 a) of the random function $V(t)$ within the interval $(-\frac{1}{2}T, \frac{1}{2}T)$, namely

$$V(t) = \sum_{n} c_n \phi_n(t). \tag{12}$$

The functions $\phi_n(t)$ are eigenfunctions of the homogeneous Fredholm integral equation

$$\lambda_n \phi_n(t) = \int_{-T/2}^{T/2} \Gamma^*(t, t') \phi_n(t') \, dt' \tag{13}$$

where

$$\Gamma(t, t') = \langle V^*(t) V(t') \rangle \tag{14}$$

and they form in general† a complete orthonormal set in the domain $-\frac{1}{2}T \leqslant t \leqslant \frac{1}{2}T$. The autocorrelation function $\Gamma(t, t')$ may be expressed in terms of the eigenvalues λ_n and eigenfunctions $\phi_n(t)$ as follows:

$$\Gamma(t, t') = \sum_n \lambda_n \phi_n^*(t)\phi_n(t'). \tag{15}$$

The coefficients c_n in equation (12) are random variables of zero mean, since the amplitude $V(t)$ is of zero mean. Moreover, they are statistically independent variables, i.e.

$$\langle c_n^* c_m \rangle = \lambda_n \delta_{nm} \tag{16}$$

where λ_n is the variance of the coefficient c_n. Now the transformation from the random amplitude $V(t)$ to the random coefficients c_n is linear. Hence the probability distribution of the set c_n will be the Gaussian distribution

$$p(\{c_n\}) = \prod_n \frac{1}{\pi\lambda_n} \exp\left(-\frac{|c_n|^2}{\lambda_n}\right). \tag{17}$$

If we substitute equations (12) and (2) into equation (7) and take into account equation (17), it follows that

$$G(s_1, s_2, \ldots, s_N) = \prod_l \frac{1}{\pi\lambda_l} \int \ldots \int \exp\left\{-\sum_{n,m}\left(\frac{1}{\lambda_n}\delta_{nm} + F_{nm}\right)c_n^* c_m\right\} \mathrm{d}^2\{c_r\} \tag{18}$$

where

$$F_{nm} = \sum_{i=1}^{N} s_i \alpha_i \int_{t_i}^{t_i+T_i} \phi_n^*(t)\phi_m(t)\, \mathrm{d}t \tag{19}$$

and

$$\mathrm{d}^2 c_k = \mathrm{d}(\mathrm{Re}\, c_k)\, \mathrm{d}(\mathrm{Im}\, c_k).$$

After performing the integrations in equation (18), we obtain for the generating function the relation

$$G(s_1, s_2, \ldots, s_N) = \frac{1}{|\delta_{nm} + \lambda_n F_{nm}|}. \tag{20}$$

Here $|\delta_{nm} + \lambda_n F_{nm}|$ denotes the determinant with elements $\delta_{nm} + \lambda_n F_{nm}$. It is an infinite determinant, when the set of eigenfunctions $\phi_n(t)$ is infinite. The value of this determinant does not vary, if we change the rows into columns. Furthermore, if we take into account that the generating function is a real function, so that it is equal to its complex conjugate, and also that

$$F_{nm}^* = F_{mn} \tag{21}$$

we conclude that

$$G(s_1, s_2, \ldots, s_N) = \frac{1}{|\delta_{nm} + F_{nm}\lambda_m|}. \tag{22}$$

Multiplying equation (20) by equation (22), and making use of the fact that the function $G(s_1, s_2, \ldots, s_N)$ is always positive (cf. equation (7)), we finally obtain the relation

$$G(s_1, s_2, \ldots, s_N) = \frac{1}{|D_{nm}|^{1/2}} \tag{23}$$

where

$$D_{nm} = \delta_{nm} + F_{nm}\lambda_m + \lambda_n F_{nm} + \sum_l \lambda_n F_{nl} F_{lm} \lambda_m. \tag{24}$$

† The autocorrelation function is a positive definite function. We assume also that it is square integrable. In this case (Davenport and Root 1958 b) the set $\{\phi_n(t)\}$ is complete in the domain $-\frac{1}{2}T \leqslant t \leqslant \frac{1}{2}T$.

In general, it is rather difficult to evaluate the generating function either from equation (20) or from equation (23), when the determinants in these equations are infinite. But it is easy to verify that the matrix with elements D_{nm} is Hermitian. It can therefore be diagonalized by a unitary transformation and the infinite matrix in equation (23) can be transformed into an infinite product.

In order to illustrate the above statement let us define the unitary matrix

$$S_{nm} = \int_{-T/2}^{T/2} \psi_n^*(t)\phi_m(t)\, dt \tag{25}$$

where $\psi_n(t)$ are eigenfunctions of the homogeneous Fredholm integral equation of the second kind

$$\mu_n\psi_n(t) = \int_{-T/2}^{T/2} K^*(t, t')\psi_n(t')\, dt'. \tag{26}$$

The Hermitian kernel $K(t, t')$ is defined by the relation

$$\begin{aligned} K(t, t') = & \sum_{i=1}^{N} s_i\alpha_i\{\Theta_i(t)+\Theta_i(t')\}\Gamma(t, t') \\ & + \sum_{i=1}^{N} (s_i\alpha_i)^2 \int_{t_i}^{t_i+T_i} \Gamma(t, \tau)\Gamma(\tau, t')\, d\tau \\ & +2\sum_{i=1}^{N}\sum_{j>i} s_is_j\alpha_i\alpha_j \int_{t_i}^{t_i+T_i} \Gamma(t, \tau)\Gamma(\tau, t')\Theta_j(\tau)\, d\tau \end{aligned} \tag{27}$$

where

$$\Theta_i(t) = \begin{matrix} 1 & \text{if } t_i < t < t_i+T_i \\ 0 & \text{otherwise.} \end{matrix} \tag{28}$$

We assume here that the kernel $K(t, t')$ is such that the eigenfunctions $\psi_n(t)$ form a complete orthonormal set in the domain $-\frac{1}{2}T \leqslant t \leqslant \frac{1}{2}T$. In this case the matrix S_{nm} is certainly unitary.

Now direct computation gives the relation

$$\sum_{r,s} S_{nr}D_{rs}S_{ms}^* = (1+\mu_n)\delta_{nm} \tag{29}$$

where μ_n are the eigenvalues of the integral equation (26). In deriving equation (29) use has been made of equations (15) and (26), the orthonormality property of the eigenfunctions $\psi_n(t)$ and the completeness property of the set $\{\phi_n(t)\}$ in the domain $-\frac{1}{2}T \leqslant t \leqslant \frac{1}{2}T$. Hence the unitary S matrix diagonalizes the Hermitian D matrix, and the diagonal elements are determined by the eigenvalues of the integral equation (26).

Taking into account the unitarity of the S matrix, it follows from equation (29) that the determinant $|D_{nm}|$ is equal to

$$|D_{nm}| = |S_{nr}||D_{rs}||(S^\dagger)_{sm}| = |\sum_{rs} S_{nr}D_{rs}S_{ms}^*| = \prod_n (1+\mu_n). \tag{30}$$

Hence the generating function in equation (23) becomes

$$G(s_1, s_2, \ldots, s_N) = \left\{\prod_n \left(\frac{1}{1+\mu_n}\right)\right\}^{1/2} \tag{31}$$

and we conclude that the eigenvalues of the integral equation (26) with the Hermitian kernel $K(t, t')$ given in equation (27) determine completely the generating function in the case of N photodetectors.

The evaluation of the eigenvalues of equation (26) with the kernel given by equation (27) is difficult in practice. Part of the difficulty is due to the fact that, although the auto-correlation function $\Gamma(t, t')$ depends on the time difference $t-t'$ for a stationary radiation field, this is not the case with the kernel $K(t, t')$.

We wish to show now how the generating function may be evaluated without the actual determination of the eigenvalues μ_n in equation (26)†. Namely, we shall show that the generating function is given by the expression

$$G(s_1, s_2, \dots, s_N) = \exp\left\{-\tfrac{1}{2}\int_0^1 \mathrm{d}\lambda \int_{-T/2}^{T/2} \mathrm{d}t\, G(s_1, s_2, \dots, s_N; t, t; \lambda)\right\}. \tag{32}$$

Here $G(s_1, \dots, s_N; t, t'; \lambda)$ is the solution of the inhomogeneous Fredholm integral equation of the second kind

$$G(s_1, \dots, s_N; t, t'; \lambda) + \lambda\int_{-T/2}^{T/2} K(t, t'')G(s_1, \dots, s_N; t'', t'; \lambda)\,\mathrm{d}t'' = K(t, t'). \tag{33}$$

The kernel $K(t, t')$ is the same as in equation (26). It is given by equation (27). We should not expect then that the problem of solving this inhomogeneous integral equation has become any simpler than that of determining the eigenvalues μ_n in equation (26).

To show the validity of the relations (32) and (33), let us define the function

$$G(s_1, \dots, s_N; t, t'; \lambda) = \sum_n \frac{\mu_n}{1+\lambda\mu_n}\psi_n^*(t)\psi_n(t') \tag{34}$$

where μ_n, $\psi_n(t)$ are the eigenvalues and eigenfunctions of equation (26). Since the set $\{\psi_n(t)\}$ is normalized, it follows that

$$\int_{-T/2}^{T/2} G(s_1, \dots, s_N; t, t; \lambda)\,\mathrm{d}t = \sum_n \frac{\mu_n}{1+\lambda\mu_n} \tag{35}$$

and hence

$$\int_0^1 \mathrm{d}\lambda \int_{-T/2}^{T/2} \mathrm{d}t\, G(s_1, \dots, s_N; t, t; \lambda) = \sum_n \ln(1+\mu_n) = -\ln\left\{\prod_n\left(\frac{1}{1+\mu_n}\right)\right\}. \tag{36}$$

If we compare this relation with the expression (31) of the generating function, equation (32) follows.

Starting now with equation (26) and using the completeness property of the set $\{\psi_n(t)\}$, we can easily show that

$$K(t, t') = \sum_n \mu_n\psi_n^*(t)\psi_n(t') \tag{37}$$

or

$$K(t, t') = \sum_n \frac{\mu_n}{1+\lambda\mu_n}\psi_n^*(t)\psi_n(t') + \lambda\sum_n \frac{\mu_n^2}{1+\lambda\mu_n}\psi_n^*(t)\psi_n(t'). \tag{38}$$

If we substitute $\mu_n\psi_n(t)$ from equation (26) into the second term in the right-hand side of equation (38) and take into account the definition (34) of $G(s_1, \dots, s_N; t, t'; \lambda)$, the integral equation (33) follows. Hence the validity of the relations (32) and (33) has been established.

We conclude then that the problem of the evaluation of the joint probability distributions or the moments for N photodetectors in the presence of a homogeneous radiation field can be reduced to the solution of either a homogeneous or an inhomogeneous Fredholm integral equation of the second kind.

There is one special case for which the generating function as given by equation (31) can be considerably simplified. It is a trivial matter to establish the identity

$$\begin{aligned}\tilde{K}(t, t') &\equiv \delta(t-t') + K(t, t') \\ &= \int_{-T/2}^{T/2} \tilde{F}(t, t'')\tilde{F}^*(t', t'')\,\mathrm{d}t''\end{aligned} \tag{39}$$

† We are indebted to Dr. G. Bedard for bringing to our attention this possibility.

where

$$\tilde{F}(t, t') \equiv \delta(t-t') + \sum_{i=1}^{N} s_i\alpha_i\Theta_i(t')\Gamma(t, t') \quad (40)$$

and $\delta(t-t')$ is the Dirac delta function. The function $\tilde{F}(t, t')$ is Hermitian in the special case, when all N photodetectors are 'open' within the same time interval $(-\frac{1}{2}T, \frac{1}{2}T)$ and only then. In this case let $1+\nu_n$ be the eigenvalues of the homogeneous integral equation

$$(1+\nu_n)\chi_n(t) = \int_{-T/2}^{T/2} \tilde{F}^*(t, t')\chi_n(t')\,\mathrm{d}t'. \quad (41)$$

From this relation and equation (39) it follows that

$$(1+\nu_n)^2\chi_n(t) = \int_{-T/2}^{T/2} \tilde{K}^*(t, t')\chi_n(t')\,\mathrm{d}t'. \quad (42)$$

Use has been made of the hermiticity of $\tilde{F}(t, t')$ in deriving this last relation. When it is compared with equation (26), it gives the expression

$$1+\mu_n = (1+\nu_n)^2. \quad (43)$$

In this case then the generating function (equation (31)) becomes

$$G(s_1, s_2, \ldots, s_N) = \prod_n \left(\frac{1}{1+\nu_n}\right) \quad (44)$$

where ν_n are the eigenvalues of the integral equation (cf. equations (40) and (41))

$$\nu_n\chi_n(t) = s\int_{-T/2}^{T/2} \Gamma^*(t, t')\chi_n(t')\,\mathrm{d}t'. \quad (45)$$

Here

$$s = \sum_{i=1}^{N} s_i\alpha_i. \quad (46)$$

The relations (44) and (45) are identical with those obtained by Slepian (1958) and used by Bedard (1966) in the case of one photodetector. The latter has actually determined the eigenvalues in equation (45) for narrow-band Gaussian light with Lorentzian spectral profile, i.e. when

$$\Gamma(t, t') = I\exp(-\Gamma|t-t'|). \quad (47)$$

He has also evaluated in this case the generating function (equation (44)), which is given by the expression

$$G(s) = \frac{\exp(\Gamma T)}{\cosh z + \sinh z(\Gamma T/2z + z/2\Gamma T)} \quad (48)$$

where

$$z = (2\Gamma T I s + \Gamma^2 T^2)^{1/2}. \quad (49)$$

In our case the parameter s is given by equation (46).

We see then that when all N photodetectors are 'open' within the same time interval and the Gaussian light has a Lorentzian spectral profile, the N-fold joint photocount distribution and the photocount moments can be explicitly evaluated by substituting equations (48), (49) and (46) into equations (5) and (6). In the general case, when the N photodetectors are 'open' within different time intervals of any magnitude and the spectral profile of the Gaussian light is arbitrary, the generating function is given by equation (31) and the eigenvalues μ_n are determined from equations (26) and (27). These relations for the generating function and the eigenvalues constitute a non-trivial generalization of the relations (44) and (45).

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